How many vector constants of motion exist for a particle moving in a central potential

1994 J. Phys. A: Math. Gen. 27 2615

(http://iopscience.iop.org/0305-4470/27/7/036)

J. Phys. A: Math. Gen. **27** (1994) 2615–2617. Printed in the UK

COMMENT

How many vector constants of motion exist for a particle moving in a central potential?

A Holas† and N H March‡

† Institute of Physical Chemistry, Polish Academy of Sciences, Kasprzaka 44/52, 01-224 Warsaw, Poland

‡ Theoretical Chemistry Department, University of Oxford, 5 South Parks Road, Oxford OX1 3UB, UK

Received 26 August 1993, in final form 4 February 1994

Abstract. Yan (1991 *J. Phys. A: Math. Gen.* **24** 4731) has drawn the conclusion that, for central fields, there exists another vector constant of motion $\mathcal{J}$, in addition to the angular momentum $\boldsymbol{L}$. It is proved here that $\dot{\mathcal{J}}$ is in fact only equal to zero 'almost everywhere'; $\mathcal{J}$ is not, therefore, rigorously a constant of motion.

As is well known, the classical Kepler problem of motion in a central, bare Coulomb potential, possesses two vector constants (integrals) of motion: the angular momentum $\boldsymbol{L}$ and the Laplace–Runge–Lenz vector $\boldsymbol{A}$. Can this property be generalized to the case of other central potentials? This question has been addressed again in the recent study of Yan [1], despite a substantial number of investigations in the last few decades devoted to this problem (see, e.g., [2] and references in both [1] and [2]). Yan concludes that, besides the angular momentum vector $\boldsymbol{L}$, there is always another vector constant of motion, $\mathcal{J}$ say, explicitly constructed in his work, and that 'the existence of this general solution does not depend on any condition, such as the orbit must be closed or that there is some hidden higher symmetry'.

The purpose of the present comment is to check whether Yan's vector $\mathcal{J}(\boldsymbol{r}, \dot{\boldsymbol{r}})$ is really a constant of motion. To do this (see also [2]), the time evolution of $\mathcal{J}$ will be studied, by its dependence on $\boldsymbol{r}(t)$ and $\dot{\boldsymbol{r}}(t)$ which are, in turn, both obtained by integration of the equation of motion: $\mathcal{J}(t) = \mathcal{J}(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t))$.

We need first to determine the limits of integration left undefined by Yan in his expression for the vector $\mathcal{J}$. First, let us introduce the potential energy instead of an integral of force (see (2.5), (2.6) and (4.1) of [1]):

$$V(r) = V(x^{1/2}) = -\tfrac{1}{2}\int_{x_0}^{x} f(s)\,\mathrm{d}s \qquad (1)$$

where the variable x, as defined by Yan,

$$x = \boldsymbol{r}\cdot\boldsymbol{r} = r^2 \qquad (2)$$

is employed. Although the choice of a particular x_0 shifts the scale of the particle energy $E = H$, this does not affect the results, because the two quantities H and V are used only in the combination $H - V$. Next, in the expression for the angular coordinate θ (equation (4.1) of [1]), we choose r_{p}^2 for the lower integration limit there:

$$\theta(x) = \frac{L}{2}\int_{r_{\mathrm{p}}^2}^{x} F^{-1/2}(s)s^{-1}\,\mathrm{d}s. \tag{3}$$

Here r_{p} is the distance from the potential centre to a perihelion of the orbit (similarly, the notation r_{a} will be adopted for the aphelion case). The above choice fixes the position of the particle on the orbit at the initial time to be at the perihelion.

To analyse in detail the expression for $\mathcal{J}$ obtained by Yan, we first note that the time derivative of x in (2) is given by

$$\dot{x} = 2\dot{\boldsymbol{r}}\cdot\boldsymbol{r} = \sigma 2|\dot{\boldsymbol{r}}\cdot\boldsymbol{r}| = 2\dot{r}r = \sigma 2|\dot{r}|r \tag{4}$$

where we have introduced the following function of the canonical variables

$$\sigma = \sigma(\boldsymbol{r}, \dot{\boldsymbol{r}}) = \mathrm{sgn}(\dot{\boldsymbol{r}}\cdot\boldsymbol{r}) = \mathrm{sgn}(\dot{r}). \tag{5}$$

This quantity equals $+1$ during each time interval when the particle moves from a perihelion to the nearest aphelion (because $r(t)$ increases) and -1 when it moves from an aphelion to the nearest perihelion (because $r(t)$ decreases). It appears that Yan did not take account of these changes in the sign of $\dot{x}$, thereby using expression (4) but with σ replaced by unity. Therefore, the right-hand side of, for instance, (2.11)–(2.13), (2.19) and (4.3) must be multiplied by σ. But (2.14)–(2.18) remain correct, because $\sigma^2 = 1$.

Now we claim that one of the correct solutions for the vector $\mathcal{J}$, satisfying the appropriate differential equations derived by Yan [1], coincides with the perihelion vector (Fradkin's vector) $\hat{\boldsymbol{k}}$ discussed in [2]:

$$\hat{\mathcal{J}}(\boldsymbol{r}, \dot{\boldsymbol{r}}) = \hat{\boldsymbol{k}}(\boldsymbol{r}, \dot{\boldsymbol{r}}) \tag{6}$$

where $\hat{\boldsymbol{a}} = \boldsymbol{a}/|\boldsymbol{a}|$ denotes a unit vector. Yan's vector is defined in terms of the scalar function A as

$$\mathcal{J} = A\dot{\boldsymbol{r}} - \dot{A}\boldsymbol{r}. \tag{7}$$

Below we set out some relations leading to the proof of (6). Comparing the definition of the function $\dot{r}_1(r)$ in equation (A4) of [2] in which mass $m = 1$, used by Yan, is substituted, with the definition $F(x)$ in equation (2.6) of [1], in which (1) and (2) are substituted, we find

$$F^{1/2}(r^2) = r\dot{r}_1(r) \tag{8}$$

which then leads to the further relation

$$\theta(r^2) = \phi_1(r) \tag{9}$$

when the definition of $\theta(x)$ in (3) is compared with equation (A3) of [2]. For the function A we take solution (4.2) of [1] of the differential equation (2.15) of [1], as found by Yan, transformed next as follows:

$$A \to -\sigma A. \tag{10}$$

(It should be noted that Yan actually applied such a transformed solution when he exemplified his results for the case of the Kepler, bare Coulomb, problem, writing them in the form (3.6) of [2].) The solution transformed according to (10) is also a valid solution of the differential equation (2.15) of [1] because this equation is homogeneous with respect to A, while σ behaves as a constant along the whole particle trajectory except at discrete points—perihelions and aphelions, excluded from consideration for the reasons to be discussed below. In our notation, such a solution A is

$$A = -\sigma \frac{\mathcal{J}}{L} r \sin\phi_1 \tag{11}$$

so that

$$-\dot{A} = \sigma \dot{r} \frac{\mathcal{J}}{L}\left(\sin\phi_1 + r\frac{\mathrm{d}\phi_1}{\mathrm{d}r}\cos\phi_1\right) = \dot{r}_1 \frac{\mathcal{J}}{L}\left(\sin\phi_1 + \frac{L}{r\dot{r}_1}\cos\phi_1\right) \tag{12}$$

where (4) has been used in the second step. Expressing $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ according to equations (2.9) and (2.11) of [2] as

$$\boldsymbol{r} = r\boldsymbol{e} \qquad \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e} + \frac{L}{r}\boldsymbol{e}' \tag{13}$$

and substituting (11)–(13) into (7), we obtain for $\hat{\mathcal{J}}$ the same expression as in equations (2.14), (2.18) and (2.21) of [2] for $\hat{\boldsymbol{k}}$, thereby proving equation (6).

Therefore, we merely recall the properties of $\hat{\boldsymbol{k}}(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t))$ established in [2]; that $\hat{\boldsymbol{k}}$ is constant during each time interval in which the particle moves along its orbit from an aphelion, via the perihelion to the next aphelion, $\hat{\boldsymbol{k}}$ being directed from the potential centre towards that perihelion, and then, at the moment the particle passes the aphelion, $\hat{\boldsymbol{k}}$ abruptly changes its direction towards the next perihelion. So $\hat{\boldsymbol{k}}(t)$ and thus $\hat{\mathcal{J}}(t)$ are almost always constant in time, except at the moments when the particle passes aphelions. Such a property was actually imposed on $\mathcal{J}$ by Yan:

$$\dot{\mathcal{J}} = 0. \tag{14}$$

We must note, however, that the function $F^{1/2}(x)$, being non-analytic at turning points $x = r_{\mathrm{p}}^2$ and r_{a}^2, is not differentiable at these points. Therefore the differential equation (2.15) of [1], deduced from (14), which in turn determines the scalar function A of the vector $\mathcal{J}$, is defined for all points of the particle trajectory except the turning points of its orbit. Consequently, the demand (14) cannot concern these points.

By way of summary, it has to be stressed that the perihelion vector for a general central force problem is a multivalued function of the coordinates [2]. The advance of this vector can be treated either as a discontinuity in its time behaviour or as a change of a branch of the multivalued function. More specifically, the fulfilment of equation (14) almost everywhere does not permit the deduction that Yan's vector $\mathcal{J}$ is a rigorous constant of motion for an arbitrary central potential and an arbitrary orbit (i.e. arbitrary E and L). Our conclusion stated in [2] that $\hat{\boldsymbol{k}}$ (and therefore $\hat{\mathcal{J}}$) can be a constant of motion only if the necessary condition given in equation (3.1) of [2] is fulfilled therefore needs reiterating. Consequences of this condition are discussed in detail in [2].

Constructive comments made by the referee have helped to improve the presentation of this work.

References

[1] Yan C C 1991 *J. Phys. A: Math. Gen.* **24** 4731–8
[2] Holas A and March N H 1990 *J. Phys. A: Math. Gen.* **23** 735–49